

Analogies between light optics and charged-particle optics

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Abstract

The close analogy between geometrical optics and the classical theories of charged-particle beam optics have been known for a very long time. In recent years, quantum theories of charged-particle beam optics have been presented with the very expected feature of wavelength-dependent effects. With the current development of non-traditional prescriptions of Helmholtz and Maxwell optics respectively, accompanied with the wavelength-dependent effects, it is seen that the analogy between the two systems persists. A brief account of the various prescriptions and the parallel of the analogies is presented.

1 Introduction

Historically, variational principles have played a fundamental role in the evolution of mathematical models in classical physics, and many equations can be derived by using them. Here the relevant examples are Fermat's principle in optics and Maupertuis' principle in mechanics. The beginning of the analogy between geometrical optics and mechanics is usually attributed to Descartes (1637), but actually it can be traced back to Ibn Al-Haitham Alhazen (0965-1037) [1]. The analogy between the trajectory of material particles in potential fields and the path of light rays in media with continuously variable

refractive index was formalized by Hamilton in 1833. This Hamiltonian analogy lead to the development of electron optics in 1920s, when Busch derived the focusing action and a lens-like action of the axially symmetric magnetic field using the methodology of geometrical optics. Around the same time Louis de Broglie associated his now famous wavelength to moving particles. Schrödinger extended the analogy by passing from geometrical optics to wave optics through his wave equation incorporating the de Broglie wavelength. This analogy played a fundamental role in the early development of quantum mechanics. The analogy, on the other hand, lead to the development of practical electron optics and one of the early inventions was the electron microscope by Ernst Ruska. A detailed account of Hamilton's analogy is available in [2]-[4].

Until very recently, it was possible to see this analogy only between the geometrical-optic and classical prescriptions of electron optics. The reasons being that, the quantum theories of charged-particle beam optics have been under development only for about a decade [5]-[13] with the very expected feature of wavelength-dependent effects, which have no analogue in the traditional descriptions of light beam optics. With the current development of the non-traditional prescriptions of Helmholtz optics [14, 15] and the matrix formulation of Maxwell optics [16]-[20], accompanied with wavelength-dependent effects, it is seen that the analogy between the two systems persists. The non-traditional prescription of Helmholtz optics is in close analogy with the quantum theory of charged-particle beam optics based on the Klein-Gordon equation. The matrix formulation of Maxwell optics is in close analogy with the quantum theory of charged-particle beam optics based on the Dirac equation. This analogy is summarized in the table of Hamiltonians. In this short note it is difficult to present the derivation of the various Hamiltonians which are available in the references. We shall briefly consider an outline of the quantum prescriptions and the non-traditional prescriptions respectively. A complete coverage to the new field of *Quantum Aspects of Beam Physics* (**QABP**), can be found in the proceedings of the series of meetings under the same name [21].

2 Quantum Formalism

The classical treatment of charged-particle beam optics has been extremely successful in the designing and working of numerous optical devices, from electron microscopes to very large particle accelerators. It is natural, however to look for a prescription based on the quantum theory, since any physical system is quantum mechanical at the fundamental level! Such a prescription is sure to explain the grand success of the classical theories and may also help get a deeper understanding and to lead to better designing of charged-particle beam devices.

The starting point to obtain a quantum prescription of charged particle beam optics is to build a theory based on the basic equations of quantum mechanics (Schrödinger, Klein-Gordon, Dirac) appropriate to the situation under study. In order to analyze the evolution of the beam parameters of the various individual beam optical elements (quadrupoles, bending magnets, ...) along the optic axis of the system, the first step is to start with the basic time-dependent equations of quantum mechanics and then obtain an equation of the form

$$i\hbar \frac{\partial}{\partial s} \psi(x, y; s) = \widehat{\mathcal{H}}(x, y; s) \psi(x, y; s) , \quad (1)$$

where $(x, y; s)$ constitute a curvilinear coordinate system, adapted to the geometry of the system. Eq. (1) is the basic equation in the quantum formalism, called as the *beam-optical equation*; \mathcal{H} and ψ as the *beam-optical Hamiltonian* and the *beam wavefunction* respectively. The second step requires obtaining a relationship between any relevant observable $\{\langle O \rangle(s)\}$ at the transverse-plane at s and the observable $\{\langle O \rangle(s_{\text{in}})\}$ at the transverse plane at s_{in} , where s_{in} is some input reference point. This is achieved by the integration of the beam-optical equation in (1)

$$\psi(x, y; s) = \widehat{U}(s, s_{\text{in}}) \psi(x, y; s_{\text{in}}) , \quad (2)$$

which gives the required transfer maps

$$\begin{aligned} \langle O \rangle(s_{\text{in}}) \longrightarrow \langle O \rangle(s) &= \langle \psi(x, y; s) | O | \psi(x, y; s) \rangle , \\ &= \langle \psi(x, y; s_{\text{in}}) | \widehat{U}^\dagger O \widehat{U} | \psi(x, y; s_{\text{in}}) \rangle . \end{aligned} \quad (3)$$

The two-step algorithm stated above gives an over-simplified picture of the quantum formalism. There are several crucial points to be noted. The

first step in the algorithm of obtaining the beam-optical equation is not to be treated as a mere transformation which eliminates t in preference to a variable s along the optic axis. A clever set of transforms are required which not only eliminate the variable t in preference to s but also give us the s -dependent equation which has a close physical and mathematical analogy with the original t -dependent equation of standard time-dependent quantum mechanics. The imposition of this stringent requirement on the construction of the beam-optical equation ensures the execution of the second-step of the algorithm. The beam-optical equation is such that all the required rich machinery of quantum mechanics becomes applicable to the computation of the transfer maps that characterize the optical system. This describes the essential scheme of obtaining the quantum formalism. The rest is mostly mathematical detail which is inbuilt in the powerful algebraic machinery of the algorithm, accompanied with some reasonable assumptions and approximations dictated by the physical considerations. The nature of these approximations can be best summarized in the optical terminology as a systematic procedure of expanding the beam optical Hamiltonian in a power series of $|\hat{\pi}_\perp/p_0|$, where p_0 is the design (or average) momentum of beam particles moving predominantly along the direction of the optic axis and $\hat{\pi}_\perp$ is the small transverse kinetic momentum. The leading order approximation along with $|\hat{\pi}_\perp/p_0| \ll 1$, constitutes the paraxial or ideal behaviour and higher order terms in the expansion give rise to the nonlinear or aberrating behaviour. It is seen that the paraxial and aberrating behaviour get modified by the quantum contributions which are in powers of the de Broglie wavelength ($\bar{\lambda}_0 = \hbar/p_0$). The classical limit of the quantum formalism reproduces the well known Lie algebraic formalism of charged-particle beam optics [22].

3 Light Optics: Non-Traditional Prescriptions

The traditional scalar wave theory of optics (including aberrations to all orders) is based on the beam-optical Hamiltonian derived by using Fermat's principle. This approach is purely geometrical and works adequately in the scalar regime. The other approach is based on the *square-root* of the Helmholtz operator, which is derived from the Maxwell equations [22]. This

approach works to all orders and the resulting expansion is no different from the one obtained using the geometrical approach of Fermat's principle. As for the polarization: a systematic procedure for the passage from scalar to vector wave optics to handle paraxial beam propagation problems, completely taking into account the way in which the Maxwell equations couple the spatial variation and polarization of light waves, has been formulated by analyzing the basic Poincaré invariance of the system, and this procedure has been successfully used to clarify several issues in Maxwell optics [23]-[26].

In the above approaches, the beam-optics and the polarization are studied separately, using very different machineries. The derivation of the Helmholtz equation from the Maxwell equations is an approximation as one neglects the spatial and temporal derivatives of the permittivity and permeability of the medium. Any prescription based on the Helmholtz equation is bound to be an approximation, irrespective of how good it may be in certain situations. It is very natural to look for a prescription based fully on the Maxwell equations, which is sure to provide a deeper understanding of beam-optics and light polarization in a unified manner.

The two-step algorithm used in the construction of the quantum theories of charged-particle beam optics is very much applicable in light optics! But there are some very significant conceptual differences to be borne in mind. When going beyond Fermat's principle the whole of optics is completely governed by the Maxwell equations, and there are no other equations, unlike in quantum mechanics, where there are separate equations for, spin-1/2, spin-1, \dots .

Maxwell's equations are linear (in time and space derivatives) but coupled in the fields. The decoupling leads to the Helmholtz equation which is quadratic in derivatives. In the specific context of beam optics, purely from a calculational point of view, the starting equations are the Helmholtz equation governing scalar optics and for a more accurate prescription one uses the full set of Maxwell equations, leading to vector optics. In the context of the two-step algorithm, the Helmholtz equation and the Maxwell equations in a matrix representation can be treated as the 'basic' equations, analogue of the basic equations of quantum mechanics. This works perfectly fine from a calculational point of view in the scheme of the algorithm we have.

Exploiting the similarity between the Helmholtz wave equation and the Klein-Gordon equation, the former is linearized using the Feshbach-Villars procedure used for the linearization of the Klein-Gordon equation. Then the

Foldy-Wouthuysen iterative diagonalization technique is applied to obtain a Hamiltonian description for a system with varying refractive index. This technique is an alternative to the conventional method of series expansion of the radical. Besides reproducing all the traditional quasiparaxial terms, this method leads to additional terms, which are dependent on the wavelength, in the optical Hamiltonian. This is the non-traditional prescription of scalar optics.

The Maxwell equations are cast into an exact matrix form taking into account the spatial and temporal variations of the permittivity and permeability. The derived representation using 8×8 matrices has a close algebraic analogy with the Dirac equation, enabling the use of the rich machinery of the Dirac electron theory. The beam optical Hamiltonian derived from this representation reproduces the Hamiltonians obtained in the traditional prescription along with wavelength-dependent matrix terms, which we have named as the *polarization terms*. These polarization terms are very similar to the spin terms in the Dirac electron theory and the spin-precession terms in the beam-optical version of the Thomas-BMT equation [10]. The matrix formulation provides a unified treatment of beam optics and light polarization. Some well known results of light polarization are obtained as a paraxial limit of the matrix formulation [23]-[26]. The traditional beam optics is completely obtained from our approach in the limit of small wavelength, $\lambda \rightarrow 0$, which we call as the traditional limit of our formalisms. This is analogous to the classical limit obtained by taking $\hbar \rightarrow 0$, in the quantum prescriptions.

From the Hamiltonians in the Table we make the following observations: The classical/traditional Hamiltonians of particle/light optics are modified by wavelength-dependent contributions in the quantum/non-traditional prescriptions respectively. The algebraic forms of these modifications in each row is very similar. This should not come as a big surprise. The starting equations have one-to-one algebraic correspondence: Helmholtz \leftrightarrow Klein-Gordon; Matrix form of Maxwell \leftrightarrow Dirac equation. Lastly, the de Broglie wavelength, λ_0 , and λ have an analogous status, and the classical/traditional limit is obtained by taking $\lambda_0 \rightarrow 0$ and $\lambda \rightarrow 0$ respectively. The parallel of the analogies between the two systems is sure to provide us with more insights.

4 Hamiltonians in Different Prescriptions

The following are the Hamiltonians, in the different prescriptions of light beam optics and charged-particle beam optics for magnetic systems. $\hat{H}_{0,p}$ are the paraxial Hamiltonians, with lowest order wavelength-dependent contributions.

Light Beam Optics	Charged-Particle Beam Optics
Fermat's Principle $\mathcal{H} = - \{n^2(\mathbf{r}) - \mathbf{p}_\perp^2\}^{1/2}$	Maupertuis' Principle $\mathcal{H} = - \{p_0^2 - \boldsymbol{\pi}_\perp^2\}^{1/2} - qA_z$
Non-Traditional Helmholtz $\hat{H}_{0,p} =$ $-n(\mathbf{r}) + \frac{1}{2n_0}\hat{\mathbf{p}}_\perp^2$ $- \frac{i\lambda}{16n_0^3} \left[\hat{\mathbf{p}}_\perp^2, \frac{\partial}{\partial z} n(\mathbf{r}) \right]$	Klein-Gordon Formalism $\hat{H}_{0,p} =$ $-p_0 - qA_z + \frac{1}{2p_0}\hat{\boldsymbol{\pi}}_\perp^2$ $+ \frac{i\hbar}{16p_0^4} \left[\hat{\boldsymbol{\pi}}_\perp^2, \frac{\partial}{\partial z} \hat{\boldsymbol{\pi}}_\perp^2 \right]$
Maxwell, Matrix $\hat{H}_{0,p} =$ $-n(\mathbf{r}) + \frac{1}{2n_0}\hat{\mathbf{p}}_\perp^2$ $- i\lambda\beta\boldsymbol{\Sigma} \cdot \mathbf{u}$ $+ \frac{1}{2n_0}\lambda^2 w^2\beta$	Dirac Formalism $\hat{H}_{0,p} =$ $-p_0 - qA_z + \frac{1}{2p_0}\hat{\boldsymbol{\pi}}_\perp^2$ $- \frac{\hbar}{2p_0} \{ \mu\gamma\boldsymbol{\Sigma}_\perp \cdot \mathbf{B}_\perp + (q + \mu)\Sigma_z B_z \}$ $+ i\frac{\hbar}{m_0c}\epsilon B_z$

Notation

Refractive Index, $n(\mathbf{r}) = c\sqrt{\epsilon(\mathbf{r})\mu(\mathbf{r})}$

Resistance, $h(\mathbf{r}) = \sqrt{\mu(\mathbf{r})/\epsilon(\mathbf{r})}$

$\mathbf{u}(\mathbf{r}) = -\frac{1}{2n(\mathbf{r})}\nabla n(\mathbf{r})$

$\mathbf{w}(\mathbf{r}) = \frac{1}{2h(\mathbf{r})}\nabla h(\mathbf{r})$

$\boldsymbol{\Sigma}$ and β are the Dirac matrices.

$\hat{\boldsymbol{\pi}}_\perp = \hat{\mathbf{p}}_\perp - q\mathbf{A}_\perp$

μ_a anomalous magnetic moment.

ϵ_a anomalous electric moment.

$\mu = 2m_0\mu_a/\hbar$, $\epsilon = 2m_0\epsilon_a/\hbar$

$\gamma = E/m_0c^2$

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